

Yamabe Classification and Prescribed Scalar Curvature in the Asymptotically Euclidean Setting

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Abstract

We prove a necessary and sufficient condition for an asymptotically Euclidean manifold to be conformally related to one with specified nonpositive scalar curvature: the zero set of the desired scalar curvature must have a positive Yamabe invariant, as defined in the article. We show additionally how the sign of the Yamabe invariant of a measurable set can be computed from the sign of certain generalized “weighted” eigenvalues of the conformal Laplacian. Using the prescribed scalar curvature result we give a characterization of the Yamabe classes of asymptotically Euclidean manifolds. We also show that the Yamabe class of an asymptotically Euclidean manifold is the same as the Yamabe class of its conformal compactification.

1 Introduction

One formulation of the prescribed scalar curvature problem asks, for a given Riemannian manifold (M^n, g) and some function R' , is there a conformally related metric g' with scalar curvature R' ? If we define $g' = \phi^{N-2}g$ for $N := \frac{2n}{n-2}$, this is equivalent to finding a positive solution of

$$-a\Delta\phi + R\phi = R'\phi^{N-1}, \quad (1.1)$$

where $a := \frac{4(n-1)}{n-2}$, R is the scalar curvature of g , and $-a\Delta + R$ is the conformal Laplacian.

On a compact manifold the Yamabe invariant of the conformal class of g poses an obstacle to the solution of (1.1). For example, in the case where M is connected and R' is constant, problem (1.1) is known as the Yamabe problem, and it admits a solution if and only if the sign of the Yamabe invariant agrees with the sign of R' [Yam60][Tru68][Aub76][Sch84]. More generally, if R' has constant sign, we can conformally transform to a metric with scalar curvature R' only if the sign of the Yamabe invariant agrees with the sign of the scalar curvature. Hence it is natural to divide conformal classes into three types, Yamabe positive, negative, and null, depending on the sign of the Yamabe invariant.

We are interested in solving equation (1.1) on a class of complete Riemannian manifolds that, loosely speaking, have a geometry approximating Euclidean space at infinity. These asymptotically Euclidean (AE) manifolds also possess a Yamabe invariant, but the relationship between the Yamabe invariant and problem (1.1) is not well understood in the AE setting, except for some results concerning Yamabe positive metrics. We have the following consequences of [Max05b] Proposition 3.

1. An AE metric can be conformally transformed to an AE metric with zero scalar curvature if and only if it is Yamabe positive. As a consequence, since the scalar curvature of an AE metric decays to zero at infinity, only Yamabe positive AE metrics can be conformally transformed to have constant scalar curvature.
2. Yamabe positive AE metrics have conformally related AE metrics with everywhere positive scalar curvature, and conformally related AE metrics with everywhere negative scalar curvature.
3. If an AE metric admits a conformally related metric with non-negative scalar curvature, then it is Yamabe positive.

Note that it was originally believed that transformation to zero scalar curvature is possible if and only if the manifold is Yamabe non-negative [CB81]. The proof in [CB81] contains an error, and the statement and proof were corrected in [Max05b]. See also [Fri11], which shows that there exists a Yamabe-null AE manifold and hence the hypotheses of [CB81] and [Max05b] are genuinely different.

As a consequence of these three facts, the situation on an AE manifold is somewhat different from the compact setting. In particular, although positive scalar curvature is a hallmark of Yamabe positive metrics, negative scalar curvature does not characterize Yamabe-negative metrics. Indeed, we show in this article that given an AE metric g , and a strictly negative function R' that decays to zero suitably at infinity, the conformal class of g includes a metric with scalar curvature R' regardless of the sign of the Yamabe invariant. So every strictly negative scalar curvature is attainable for every conformal class, but zero scalar curvature is attainable only for Yamabe positive metrics. Thus we are lead to investigate the role of the Yamabe class in the boundary case of prescribed non-positive scalar curvature.

Rauzy treated the analogous problem on smooth compact Riemannian manifolds in [Rau95], which contains the following statement. Suppose $R' \leq 0$ and $R' \not\equiv 0$. Observe that if R' is the scalar curvature of a metric conformally related to some g , then g must be Yamabe-negative, and without loss of generality we assume that g has constant negative scalar curvature R . Then there is a metric in the conformal class of g with scalar curvature R' if and only if

$$a\lambda_{R'} > -R \tag{1.2}$$

where a is the constant from equation (1.1) and where

$$\lambda_{R'} = \inf \left\{ \frac{\int |\nabla u|^2}{\int u^2} : u \in W^{1,2}, u \geq 0, u \not\equiv 0, \int R' u = 0 \right\}. \tag{1.3}$$

Rauzy's condition (1.2) is not immediately applicable on asymptotically Euclidean manifolds, in part because of the initial transformation to constant negative scalar curvature. However, recalling that R is constant we can write $a\lambda_{R'} + R$ as the infimum of

$$\frac{\int a|\nabla u|^2 + Ru^2}{\int u^2} \tag{1.4}$$

over functions u supported in the region where $R' = 0$. So, morally, inequality (1.2) expresses the positivity of the first eigenvalue of the conformal Laplacian of the constant scalar curvature metric g on the region $\{R' = 0\}$. The connection between the first eigenvalue of the conformal Laplacian and prescribed scalar curvature problems is well known, but its use is more technical on non-compact manifolds where true eigenfunctions need not exist. For example, [FCS80] shows that a metric on a noncompact manifold can be conformally

transformed to a scalar flat one if and only if the first eigenvalue of the conformal Laplacian is positive on every bounded domain.

In this article we extend these ideas in a number of ways to solve the prescribed non-positive scalar curvature problem on asymptotically Euclidean manifolds, and we obtain a related characterization of the Yamabe class of an AE metric. In particular, we show the following.

- Every measurable subset $V \subseteq M$ can be assigned a number $y(V)$ that generalizes the Yamabe invariant of a manifold. The invariant depends on the conformal class of the AE metric, but is independent of the conformal representative.
- We can assign every measurable subset $V \subseteq M$ a number $\lambda_\delta(V)$ that generalizes the first eigenvalue of the conformal Laplacian. These numbers are not conformal invariants, and are not even canonically defined as they depend on a choice of parameters (a number δ and a choice of weight function at infinity). Nevertheless the sign of $\lambda_\delta(V)$ agrees with the sign of $y(V)$, regardless of the choice of these parameters.
- Given a candidate scalar curvature $R' \leq 0$, there is a metric in the conformal class of g with scalar curvature R' if and only if $\{R' = 0\}$ is Yamabe positive, i.e., $y(\{R' = 0\}) > 0$.
- A metric is Yamabe positive if and only if every scalar curvature $R' \leq 0$ is attained by a metric in its conformal class.
- A metric is Yamabe null if and only if every scalar curvature $R' \leq 0$, except for $R' \equiv 0$, is attained by a metric in its conformal class.
- A metric is Yamabe negative if and only if there is a scalar curvature $R' \leq 0$, $R' \not\equiv 0$, that is unattainable within the conformal class. We also present some results concerning which scalar curvatures have Yamabe positive zero sets, and hence are attainable.
- Additionally, a metric is Yamabe positive/negative/null if and only if it admits a conformal compactification to a metric with the same Yamabe type.

These results carry over to compact manifolds, where we obtain some technical improvements. First, Rauzy's condition (1.2) is equivalent to our condition $y(\{R' = 0\}) > 0$ (or equivalently $\lambda_\delta(\{R' = 0\}) > 0$). But the condition $y(\{R' < 0\}) > 0$ can be measured without reference to a particular background metric. Moreover, we work with fairly general metrics ($W_{\text{loc}}^{2,p}$ with $p > n/2$), and arbitrary scalar curvatures in $L^p(M)$. Finally, there is an error in Rauzy's proof, closely related to the gap in Yamabe's original attempt at the Yamabe problem, that we correct in our presentation.¹

The prescribed scalar curvature problem on AE manifolds when $R' \geq 0$, or when R' changes sign, remains open. Of course if $R' \geq 0$ the problem can only be solved if the manifold is Yamabe positive, but it is not known the extent to which this is sufficient. For scalar curvatures that change sign, little is known for any Yamabe class. Nevertheless, the case $R' \leq 0$ that we treat here has an interesting application to general relativity; see below. For comparison, we note that the prescribed scalar curvature problem on a compact manifold is also not yet fully solved. On a Yamabe-positive manifold it is necessary that $R' > 0$ somewhere, and on a Yamabe-null manifold it is necessary that either $R' \equiv 0$, or $R' > 0$ somewhere and $\int R' < 0$

¹We would like to thank Rafe Mazzeo for having spotted our own error in this regard while this work was in preparation.

when computed with respect to the scalar flat conformal representative. See [ES86] which shows that these conditions are sufficient in some cases. See also [BE87] for obstructions posed by conformal Killing fields.

Our interest in this problem stems from its application to general relativity. Initial data for the Cauchy problem must satisfy certain compatibility conditions known as the Einstein constraint equations. One approach to finding solutions of the constraint equations, the so-called conformal method, involves solving a coupled system of PDEs that includes the Lichnerowicz equation, which in the vacuum case is

$$-a\Delta\phi + R\phi + \frac{n-1}{n}\tau^2\phi^{N-1} - \beta^2\phi^{-N-1} = 0. \quad (1.5)$$

Here ϕ is an unknown conformal factor, τ is a prescribed function (a mean curvature, in fact), and, for the discussion at hand, β can be thought of as a prescribed function as well. On a compact Yamabe-negative manifold, the Lichnerowicz equation (1.5) is solvable if and only if the prescribed scalar curvature problem (1.1) is solvable for $R' = -\tau^2$ [Max05a]. An analogous condition holds on AE manifolds [DGI15], and hence the prescribed non-negative scalar curvature problem is intimately connected to the solvability of the Lichnerowicz equation. In particular, on an AE or Yamabe-negative compact manifold, the Lichnerowicz equation can only be solved if the zero set of the mean curvature τ is Yamabe positive.

2 Asymptotically Euclidean Manifolds

Throughout this article we assume that (M, g) is a connected Riemannian n -manifold. An asymptotically Euclidean (AE) manifold is a complete manifold such that for some compact $K \subset M$, the complement $M \setminus K$ has finitely many components E_i , with each E_i admitting a distinguished diffeomorphism to the exterior of a ball in \mathbb{R}^n . The E_i are called the ends of M , and in end coordinates the metric g decays at infinity to the standard Euclidean metric e .

In order to make this notion precise we use weighted Sobolev spaces. Let $\rho \geq 1$ be a smooth function on M that agrees with the Euclidean radial coordinate function near infinity on each end, and let \hat{g} be a smooth metric on M that equals the Euclidean metric in a neighborhood of each infinity. We say that a L^1_{loc} tensor T belongs to $W^{k,p}_\delta(M)$ if

$$\|T\|_{W^{k,p}_\delta(M)} := \sum_{j=0}^k \|\rho^{-\delta-\frac{n}{p}+j} |\nabla^j T|\|_{L^p(M)} < \infty \quad (2.1)$$

where all metric quantities in equation (2.1) use \hat{g} . When $k = 0$, we denote the space by $L^p_\delta(M)$ with norm $\|\cdot\|_{p,\delta}$. It is easy to see that the spaces $W^{k,p}_\delta$ are independent of the choice of background metric \hat{g} , and that the associated norms are equivalent. There are varying conventions in the literature for the weight parameter δ in equation (2.1), and we follow [Bar86]. Consequently, functions in $W^{k,p}_\delta$ have asymptotic growth $O(r^\delta)$ on each end. Other properties of weighted spaces can be found in [Bar86], and they parallel those for Sobolev spaces on compact manifolds. There are two key subtleties. First, L^p_δ embeds in $L^{p'}_{\delta'}$ if $p > p'$ and $\delta < \delta'$, but this is not true if $\delta = \delta'$. Second, the embedding

$$W^{k,p}_\delta \hookrightarrow W^{k-1,p'}_{\delta'} \quad (2.2)$$

is compact so long both

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{p'} \quad \text{and} \quad \delta < \delta'. \quad (2.3)$$

We also have Sobolev embedding into spaces of continuous functions. A function u belongs to $L^\infty_\delta(M)$ if

$$\|u\|_{L^\infty_\delta(M)} := \sup_M |u| \rho^{-\delta} < \infty \quad (2.4)$$

and we write C^0_δ for the continuous elements of L^∞_δ . Then $W^{k,p}_\delta \subset C^0_\delta$ for $p > n/k$ [Bar86].

We say that g is a $W^{k,p}_\tau$ AE metric if $\tau < 0$ and

$$g - \hat{g} \in W^{k,p}_\tau. \quad (2.5)$$

We will work exclusively with $W^{2,p}_\tau$ AE metrics with $p > n/2$, and we henceforth assume

$$p > n/2 \quad \text{and} \quad \tau < 0. \quad (2.6)$$

A $W^{2,p}_\tau$ metric is Hölder continuous and has curvatures in $L^p_{\tau-2}$. Using the fact that $W^{2,p}_{\text{loc}}$ is an algebra, a straightforward computation shows that we can use a $W^{2,p}_\tau$ metric for the metric quantities in equation (2.1) to obtain an equivalent norm, so long as $0 \leq k \leq 2$. We will use this definition of the norm whenever it is appropriate.

The Laplacian and conformal Laplacian of a $W^{2,p}_\tau$ metric are well-defined as maps from $W^{2,q}_\delta$ to $L^q_{\delta-2}$ for $q \in (1, p]$, they are Fredholm with index 0 if $\delta \in (2 - n, 0)$, and indeed the Laplacian is an isomorphism in this range; see, e.g., [Bar86] Proposition 2.2. Note that [Bar86] works on a manifold diffeomorphic to \mathbf{R}^n , but the results we cite from [Bar86] extend to manifolds with general topology and any finite number of ends.

Many of the results in this article hold for both asymptotically Euclidean and compact manifolds, and indeed we can often treat a $W^{2,p}$ metric on a compact manifold as a $W^{2,p}_\tau$ metric on an asymptotically Euclidean manifold with zero ends, in which case the weight function ρ is irrelevant and could be set to 1 if desired. For the sake of brevity, throughout Section 3 we interpret a compact manifold as an AE manifold with zero ends. In the remaining sections there are differences between the two cases and we assume that AE manifolds have at least one end.

The weight parameter

$$\delta^* = \frac{2 - n}{2} \quad (2.7)$$

plays a prominent role in this paper, and it reflects the minimum decay needed to ensure $\int |\nabla u|^2$ is finite. At this decay rate, $L^N_{\delta^*} = L^N$ and we have the following inequalities that generalize the Poincaré and Sobolev inequalities on \mathbf{R}^n .

Lemma 2.1. *Let (M, g) be a non-compact $W^{2,p}_\tau$ AE manifold. There exists constants c_1, c_2 such that*

$$\|\nabla u\|_2 \geq c_1 \|u\|_{2, \delta^*} \quad (2.8)$$

$$\|\nabla u\|_2 \geq c_2 \|u\|_N \quad (2.9)$$

for all $u \in W^{1,2}_{\delta^*}(M)$, where δ^* is defined in equation (2.7) and where N is the critical Sobolev exponent $2n/(n-2)$.

Proof. Suppose to the contrary that we can find a sequence $\{u_k\}$ of smooth functions with $\|u_k\|_{2,\delta^*} = 1$ and $\|\nabla u_k\|_2 \rightarrow 0$. It then follows that $\{u_k\}$ is bounded in $W_{\delta^*}^{1,2}(M)$ and therefore a subsequence (which we reduce to) converges to a weak limit $u \in W_{\delta^*}^{1,2}(M)$. Since $\nabla u_k \rightarrow 0$ in L^2 we conclude that u is constant, and since $\delta^* < 0$ we conclude that $u = 0$. Moreover, $u_k \rightarrow 0$ strongly in L^2 on compact sets.

Let η be a cutoff function that equals 1 outside of some large ball and has support contained in the ends of M . Since $\nabla u_k \rightarrow 0$ in $L^2(M)$ and since $u_k \rightarrow 0$ in L^2 on compact sets we see that $\nabla(\eta u_k) \rightarrow 0$ in $L^2(M)$. Also, since $u_k \rightarrow 0$ in L^2 on compact sets it follows that $(1 - \eta)u_k \rightarrow 0$ in $L^2_{\delta^*}$. Since $\|u_k\|_{2,\delta^*} = 1$ and since $\|(1 - \eta)u_k\|_{2,\delta^*} \rightarrow 0$ it follows that $\|\eta u_k\|_{2,\delta^*} \rightarrow 1$.

From the weighted Poincaré inequality [Bar86] Theorem 1.3(ii) we know that there is a constant c such that for all k ,

$$\|\eta u_k\|_{\bar{g},2,\delta^*} \leq c \|\nabla(\eta u_k)\|_{\bar{g},2} \quad (2.10)$$

where \bar{g} is the Euclidean metric on the end. But g is comparable to \bar{g} on the end, so this same inequality holds for g after suitably modifying c . This is a contradiction.

The proof of inequality (2.9) is essentially the same as (2.8). \square

Lemma 2.1 fails on compact manifolds due to the presence of the constants. For our proofs that treat the compact and non-compact case simultaneously it will be helpful to have a suitable inequality that works in both settings. Observe that for any $\delta > 0$ there exists c_2 such that

$$\|u\|_{2,\delta} + \|\nabla u\|_2 \geq c_2 \|u\|_N. \quad (2.11)$$

This follows from the standard Sobolev inequality on compact manifolds and follows trivially from inequality (2.9) on non-compact manifolds.

3 The Yamabe Invariant of a Measurable Set

Throughout this section, let (M, g) be a $W_\tau^{2,p}$ AE manifold with $p > n/2$ and $\tau < 0$, with the convention that a compact manifold is an AE manifold with zero ends. For $u \in C_c^\infty(M)$, $u \neq 0$, the Yamabe quotient of u is

$$Q_g^y(u) = \frac{\int a |\nabla u|^2 + R u^2}{\|u\|_N^2} \quad (3.1)$$

and the Yamabe invariant of g is the infimum of Q_g^y taken over $C_c^\infty(M)$. Here and in other notations we will drop the decoration g when the metric is understood. Our principal goal in this section is to define a similar conformal invariant for arbitrary measurable subsets of M and to analyze its properties.

It will be convenient to work with a complete function space, and we claim that the domain of Q^y can be extended to $W_{\delta^*}^{1,2} \setminus \{0\}$ where δ^* is defined in equation (2.7). To see this, first note from the embedding properties of weighted Sobolev spaces that $W_{\delta^*}^{1,2}$ embeds continuously in $L^N = L_{\delta^*}^N$ and that $u \mapsto \nabla u$ is continuous from $W_{\delta^*}^{1,2}$ to L^2 ; indeed δ^* is the minimum decay needed to ensure these conditions. To treat the scalar curvature term in Q^y , we have the following.

Lemma 3.1. *The map*

$$u \mapsto \int Ru^2 \quad (3.2)$$

is weakly continuous on $W_{\delta^}^{1,2}$. Moreover, for any $\delta > \delta^*$ and $\epsilon > 0$, there is constant $C > 0$ such that*

$$\left| \int Ru^2 \right| \leq \epsilon \|\nabla u\|_2^2 + C \|u\|_{2,\delta}^2. \quad (3.3)$$

Proof. Recall that $R \in L_{\tau-2}^p$ where $p > n/2$ and $\tau < 0$. So there is an $s \in (0, 1)$ such that

$$\frac{1}{p} = s \frac{2}{n}. \quad (3.4)$$

Set $\sigma = \delta^* - \tau/2$. Since $s < 1$ and $\sigma > \delta^*$, $W_{\delta^*}^{1,2}$ embeds compactly in $W_{\sigma}^{s,2}$, where the interpolation space $W_{\sigma}^{s,2}$ is described in [Tri76a][Tri76b]. Moreover, $W_{\sigma}^{s,2}$ embeds continuously in L_{σ}^q where

$$\frac{1}{q} = \frac{1}{2} - \frac{s}{n} = \frac{1}{2} \left(1 - \frac{1}{p} \right). \quad (3.5)$$

Since

$$\frac{1}{p} + \frac{2}{q} = 1 \quad (3.6)$$

and since

$$\tau - 2 + 2\sigma = 2\delta^* - 2 = -n, \quad (3.7)$$

Hölder's inequality implies the map (3.2) is continuous on L_{σ}^q , and from the previously mentioned compact embedding the map (3.2) is therefore weakly continuous on $W_{\delta^*}^{1,2}$. Moreover, Hölder's inequality implies there is a constant C such that

$$\left| \int Ru^2 \right| \leq C \|u\|_{W_{\sigma}^{s,2}}^2. \quad (3.8)$$

From interpolation [Tri76b] we have

$$\|u\|_{W_{\sigma}^{s,2}} \leq C \|u\|_{W_{\delta^*}^{1,2}}^s \|u\|_{2,\delta}^{1-s} \quad (3.9)$$

where δ satisfies

$$s\delta^* + (1-s)\delta = \sigma. \quad (3.10)$$

Since $\sigma = \delta^* - \tau/2$, we find

$$\delta = \delta^* - \frac{\tau/2}{1-s}, \quad (3.11)$$

and since $\tau < 0$ and $s \in (0, 1)$, $\delta > \delta^*$. Indeed, by raising τ close to zero, or lowering p close to $n/2$ (which raises s up to 1), we can obtain any particular $\delta > \delta^*$. We conclude from inequalities (3.8), (3.9) and the arithmetic-geometric mean inequality that

$$\left| \int Ru^2 \right| \leq \epsilon \|\nabla u\|_{W_{\delta^*}^{1,2}}^2 + C \|u\|_{2,\delta}^2. \quad (3.12)$$

This establishes inequality (3.3) on a compact manifold, and we obtain (3.3) in the non-compact case by applying the Poincaré inequality (2.8). \square

Corollary 3.2. *The map*

$$u \mapsto \int a|\nabla u|^2 + Ru^2 \quad (3.13)$$

is weakly upper semicontinuous on $W_{\delta^}^{1,2}$.*

Proof. This follows from the weak upper semicontinuity of $u \mapsto \int |\nabla u|^2$ along with Lemma 3.1. \square

Definition 3.3. Let $V \subseteq M$ be a measurable set. The *test functions supported in V* are

$$A(V) := \left\{ u \in W_{\delta^*}^{1,2}(M) : u \not\equiv 0, u|_{V^c} = 0 \right\}. \quad (3.14)$$

Definition 3.4. Let $V \subseteq M$ be measurable. The *Yamabe invariant* of V is

$$y_g(V) = \inf_{u \in A(V)} Q^y(u). \quad (3.15)$$

If V has measure zero, and hence $A(V)$ is empty, we use the convention $y_g(V) = \infty$.

In principle, the infimum in the definition of the Yamabe invariant could be $-\infty$. The following estimate, which will be useful later in the paper as well, shows that this is not possible.

Lemma 3.5. *Let $\delta \in \mathbb{R}$. There exist positive constants C_1 and C_2 such that for all $u \in W_{\delta^*}^{1,2}$,*

$$\|u\|_{W_{\delta^*}^{1,2}} \leq C_1 \left[\int a|\nabla u|^2 + Ru^2 \right] + C_2 \|u\|_{2,\delta}^2. \quad (3.16)$$

Proof. It is enough to establish inequality (3.16) assuming $\delta > \delta^*$. From Lemma 3.1, there is a constant C such that

$$\left| \int Ru^2 \right| \leq \frac{a}{2} \int |\nabla u|^2 + C \|u\|_{2,\delta}^2 \quad (3.17)$$

and hence

$$\int a|\nabla u|^2 + Ru^2 \geq \frac{a}{2} \int |\nabla u|^2 - C \|u\|_{2,\delta}^2. \quad (3.18)$$

Consequently

$$\int |\nabla u|^2 \leq \frac{2}{a} \left[\int a|\nabla u|^2 + Ru^2 \right] + \frac{2C}{a} \|u\|_{2,\delta}^2. \quad (3.19)$$

Inequality (3.16) now follows trivially in the compact case, and follows from the Poincaré inequality (2.8) in the non-compact case. \square

Lemma 3.6. *For every measurable set V , $y(V) > -\infty$.*

Proof. Let u_k be some minimizing sequence for Q^y normalized so that $\|u_k\|_N = 1$. Lemma 3.5 and the continuous embedding $L^N \hookrightarrow L_\delta^2$ implies that u_k is uniformly bounded in $W_{\delta^*}^{1,2}$. Estimate (3.3) then implies that $Q(u_k)$ is uniformly bounded below. \square

As one might expect, $y(V)$ is a conformal invariant.

Lemma 3.7. Suppose $g' = \phi^{N-2}g$ is a conformally related metric with $\phi - 1 \in W_\tau^{2,p}$. Then

$$y_{g'}(V) = y_g(V). \quad (3.20)$$

Proof. The conformal transformation laws

$$\begin{aligned} dV_{g'} &= \phi^N dV_g \\ R_{g'} &= \phi^{1-N}(-a\Delta_g \phi + R_g \phi) \end{aligned} \quad (3.21)$$

together with an integration by parts imply

$$\int_M |\nabla u|_{g'}^2 + R_{g'} u^2 dV_{g'} = \int_M |\nabla(\phi u)|_g^2 + R_g(\phi u)^2 dV_g \quad (3.22)$$

for all $u \in W_{\delta^*}^{1,2}(M)$. Since $\|\cdot\|_{g',N} = \|\phi \cdot\|_{g,N}$, it follows that

$$Q_{g'}^y(u) = Q_g^y(\phi u) \quad (3.23)$$

for all $u \in W_{\delta^*}^{1,2}(M)$ as well. Since $A(V)$ is invariant under multiplication by ϕ , $y_{g'}(V) = y_g(V)$. \square

We will primarily be interested in the sign of the Yamabe invariant.

Definition 3.8. A measurable set $V \subseteq M$ is called *Yamabe positive*, *negative*, or *null* depending on the sign of $y(V)$.

The Yamabe invariant involves the critical Sobolev exponent N and hence can be technically difficult to work with. On a compact manifold, however, the sign of the Yamabe invariant can be determined from the sign of the first eigenvalue of the conformal Laplacian. These eigenvalues enjoy superior analytical properties, and we now describe how to extend this approach to measurable subsets of compact or asymptotically Euclidean manifolds.

For $\delta > \delta^*$ we define the Rayleigh quotients

$$Q_{g,\delta}(u) = \frac{\int a|\nabla u|^2 + Ru^2}{\|u\|_{2,\delta}^2}. \quad (3.24)$$

Our previous arguments for the Yamabe quotient imply that $Q_{g,\delta}$ is well-defined for any $u \in W_{\delta^*}^{1,2} \setminus \{0\}$, and indeed $Q_{g,\delta}$ is continuous on this set.

Definition 3.9. The first δ -weighted eigenvalue of the conformal Laplacian is

$$\lambda_{g,\delta}(V) = \inf_{u \in A(V)} Q_{g,\delta}(u). \quad (3.25)$$

By convention, if V has measure zero then $\lambda_{g,\delta}(V) = \infty$. We will write Q_δ and λ_δ when the metric is understood.

The value of $\lambda_\delta(V)$ is not particularly meaningful; it depends on the choice of weight function ρ and it is not a conformal invariant. Nevertheless, its sign is a conformal invariant independent of the choice of ρ .

Proposition 3.10. *For any measurable set $V \subseteq M$, the following are equivalent:*

1. $y(V) > 0$.
2. $\lambda_\delta(V) > 0$ for all $\delta > \delta^*$.
3. $\lambda_\delta(V) > 0$ for some $\delta > \delta^*$.

Proof. We assume that V has positive measure since the equivalence is trivial otherwise. The implication $1 \Rightarrow 2$ follows from the inequality $\|u\|_{2,\delta} \leq C\|u\|_N$ applied to Q^y . The implication $2 \Rightarrow 3$ is trivial. So it remains to show that $3 \Rightarrow 1$.

Let V be a measurable set with $\lambda_\delta(V) > 0$ for some $\delta > \delta^*$. Suppose to produce a contradiction that $y(V) \leq 0$. Then there is a sequence $u_k \in A(V)$, normalized so that $\int a|\nabla u_k|^2 + \|u_k\|_{2,\delta}^2 = 1$, such that $Q^y(u_k) \leq 1/k$. Then

$$\lambda_\delta(V)\|u_k\|_{2,\delta}^2 \leq \int a|\nabla u_k|^2 + Ru_k^2 \leq \frac{1}{k}\|u_k\|_N^2 \leq \frac{c}{k} \left[\int a|\nabla u_k|^2 + \|u_k\|_{2,\delta}^2 \right] \leq \frac{c}{k} \quad (3.26)$$

by the Sobolev inequality (2.11). In particular, $\|u_k\|_{2,\delta}^2 \rightarrow 0$. Using inequality (3.26), we also find that

$$\int Ru_k^2 \leq \frac{c}{k} - \int a|\nabla u_k|^2 \rightarrow -1. \quad (3.27)$$

However, by Lemma 3.1, there exists $C > 0$ such that

$$\left| \int Ru_k^2 \right| \leq \frac{a}{2}\|\nabla u_k\|_2^2 + C\|u_k\|_{2,\delta}^2 \rightarrow \frac{1}{2}, \quad (3.28)$$

which is a contradiction. \square

Corollary 3.11. *For a measurable set $V \subseteq M$, the signs of $y(V)$ and $\lambda_\delta(V)$ are the same for any $\delta > \delta^*$.*

Proof. Proposition 3.10 shows that $y(V)$ is positive if and only if $\lambda_\delta(V)$ is also. Choosing an appropriate test function shows that $y(V)$ is negative if and only if $\lambda_\delta(V)$ is also. Together, these imply that $y(V)$ is zero if and only if $\lambda_\delta(V)$ is. \square

The decay rate δ^* is critical for Corollary 3.11. For $\delta < \delta^*$, $W_{\delta^*}^{1,2}$ is not contained in L_δ^2 and hence our definition of λ_δ does not extend to this range. One could minimize Q_δ over smooth functions instead to define λ_δ , but using rescaled bump functions on large balls as test functions it can be shown that $\lambda_\delta(\mathbb{R}^n) = 0$ for $\delta < \delta^*$ despite the fact that Lemma 2.1 implies $y(\mathbb{R}^n) > 0$. Note that we have not addressed equality in the threshold case $\delta = \delta^*$.

We now turn to continuity properties of λ_δ . Monotonicity is obvious from the definition.

Lemma 3.12. *Let $\delta > \delta^*$. If V_1 and V_2 are measurable sets with $V_1 \subseteq V_2$, then $\lambda_\delta(V_1) \geq \lambda_\delta(V_2)$.*

Note that Lemma 3.12 holds even for $V_1 = \emptyset$, and that this relies on our definition $\lambda_\delta(\emptyset) = y(\emptyset) = \infty$. To obtain more refined properties of λ_δ , we start by showing that minimizers of the Rayleigh quotients exist and are generalized eigenfunctions.

Proposition 3.13. *Let V be a measurable set with positive measure and let $\delta > \delta^*$. There exists a non-negative $u \in A(V)$ that minimizes Q_δ over $A(V)$. Moreover, on any open set contained in V ,*

$$-a \Delta u + Ru = \lambda_\delta(V) \rho^{2(\delta^* - \delta)} u. \quad (3.29)$$

Proof. Let u_k be a minimizing sequence in $A(V)$; this uses the hypothesis that V has positive measure. Without loss of generality we may assume that each $\|u_k\|_{2,\delta} = 1$. Since

$$a \int_M |\nabla u_k|^2 + Ru_k^2 = Q_\delta(u_k), \quad (3.30)$$

and since u_k is a minimizing sequence, Lemma 3.5 implies $\{u_k\}$ is bounded in $W_{\delta^*}^{1,2}(M)$ and hence converges weakly in $W_{\delta^*}^{1,2}(M)$ and strongly in $L_\delta^2(M)$ to a limit $u \in W_{\delta^*}^{1,2}(M)$ with $\|u\|_{2,\delta} = 1$. Since each $u_k = 0$ on V^c , from the strong L_δ^2 convergence we see $u = 0$ on V^c , and since $u \neq 0$ we conclude that $u \in A(V)$. Weak upper semicontinuity (Corollary 3.2) implies u minimizes Q_δ over the test functions $A(V)$. Noting that $|u|$ is also a minimizer, we may assume $u \geq 0$.

Suppose V contains an open set Ω . Then any $\phi \in C_c^\infty(\Omega)$ with $\phi \not\equiv 0$ belongs to $A(V)$, and we can differentiate $Q_\delta(u + t\phi)$ at $t = 0$ to find u is a weak solution in Ω of equation (3.29). \square

Lemma 3.14 (Continuity from above). *Let $V \subseteq M$ be a measurable set. If $\{V_k\}$ is a decreasing sequence of measurable sets with $\cap V_k = V$, then*

$$\lim_{k \rightarrow \infty} \lambda_\delta(V_k) = \lambda_\delta(V). \quad (3.31)$$

Proof. From the elementary monotonicity of λ_δ , $\Lambda = \lim_{k \rightarrow \infty} \lambda_\delta(V_k)$ exists and

$$\lambda_\delta(V_k) \leq \Lambda \leq \lambda_\delta(V) \quad (3.32)$$

for each k . So it is enough to show that

$$\Lambda \geq \lambda_\delta(V). \quad (3.33)$$

We may assume that Λ is finite, for inequality (3.33) is trivial otherwise. As a consequence, each V_k is nonempty and Proposition 3.13 provides minimizers u_k of Q_δ over $A(V_k)$ satisfying $\|u_k\|_{2,\delta} = 1$. For each k , since $\|u_k\|_{2,\delta} = 1$,

$$\int a |\nabla u_k|^2 + Ru_k^2 \leq \Lambda. \quad (3.34)$$

From inequality (3.34) and the boundedness of the sequence in $L_\delta^2(M)$, Lemma 3.5 implies the sequence is bounded in $W_{\delta^*}^{1,2}(M)$. A subsequence converges weakly in $W_{\delta^*}^{1,2}(M)$ and strongly in $L_\delta^2(M)$ to a limit v with $\|v\|_{2,\delta} = 1$. From weak upper semicontinuity (Corollary 3.2) we conclude $Q_\delta(v) \leq \Lambda$ as well. Moreover, $v \in A(V)$ since $v = 0$ on V_k^c . So $\lambda_\delta(v) \leq \Lambda$. \square

Note that Lemma 3.14 is false for the Yamabe invariant. For example, one can take a sequence of balls in \mathbb{R}^n that shrink down to the empty set. It is easy to see that the Yamabe invariant is scale invariant and hence is a finite constant along the sequence. Yet the Yamabe invariant of the empty set is infinite. In contrast, if $V_n \searrow \emptyset$, Lemma 3.14 implies $\lambda_\delta(V_n) \rightarrow \infty$, and in particular at some point along the sequence $\lambda_\delta(V_n) > 0$. The following result, which is an extension of [Rau95] Lemma 2 to the AE setting, shows that in fact $\lambda_\delta(V)$ is positive so long as a certain weighted volume is sufficiently small.

Lemma 3.15 (Small sets are Yamabe positive). *For any $\mu > n$, there exists $C > 0$ such that if $\text{Vol}_\mu(V) := \int_V \rho^{-\mu} < C$, V is Yamabe positive.*

Proof. Suppose that $u \in A(V)$. Define δ by $(-2\delta - n)\frac{n}{2} = -\mu$. Note that $\mu > n$ implies that $\delta > \delta^*$. Then, by Hölder's inequality,

$$\|u\|_{2,\delta}^2 = \int u^2 \rho^{-2\delta-n} \leq \left(\int u^N \right)^{2/N} \left(\int_V \rho^{(-2\delta-n)\frac{n}{2}} \right)^{2/n} = \|u\|_N^2 \text{Vol}_\mu(V)^{2/n}. \quad (3.35)$$

By the Sobolev inequality (2.11), there exists C_1 such that

$$\|u\|_N^2 \leq C_1 \left[\int a|\nabla u|^2 + \|u\|_{2,\delta}^2 \right]. \quad (3.36)$$

We also note that Lemma 3.1 implies there exists C_2 such that

$$-C_2 \|u\|_{2,\delta}^2 \leq \frac{1}{2} \int a|\nabla u|^2 + \int Ru^2. \quad (3.37)$$

Let η be defined by $\eta \text{Vol}_\mu(V)^{2/n} C_1 = \frac{1}{2}$. Using inequalities (3.35)-(3.37), we calculate

$$\begin{aligned} (\eta - C_2) \|u\|_{2,\delta}^2 &\leq \eta \|u\|_N^2 \text{Vol}_\mu(V)^{2/n} + \int Ru^2 + \frac{1}{2} \int a|\nabla u|^2 \\ &\leq \eta \text{Vol}_\mu(V)^{2/n} C_1 \left[\int a|\nabla u|^2 + \|u\|_{2,\delta}^2 \right] + \int Ru^2 + \frac{1}{2} \int a|\nabla u|^2 \\ &= \int (a|\nabla u|^2 + Ru^2) + \frac{1}{2} \|u\|_{2,\delta}^2. \end{aligned} \quad (3.38)$$

Dividing through by $\|u\|_{2,\delta}^2$, inequality (3.38) reduces to

$$\eta - C_2 - \frac{1}{2} \leq Q_\delta(u). \quad (3.39)$$

As $\text{Vol}_\mu(V) \rightarrow 0$, $\eta \rightarrow \infty$. Thus there is a $C > 0$ such that if $\text{Vol}_\mu(V) < C$, then $Q_\delta(u)$ has a uniform positive lower bound for all $u \in A(V)$. Thus $\lambda_\delta(V) > 0$, and so V is Yamabe positive by Corollary 3.11. \square

In Section 5 below we discuss the relationship between the Yamabe invariant of an AE manifold and its compactification. After compactification, for $\mu = 2n$, the condition $\text{Vol}_\mu(V) < C$ corresponds to the condition that the usual volume of the compactified set is sufficiently small. This is exactly Rauzy's condition, and the other choices of μ provide a mild generalization of his result.

Lemma 3.16 (Strict monotonicity at connected, open sets). *Let $\delta > \delta^*$ and let Ω be a connected open set. For any measurable set E in Ω with positive measure,*

$$\lambda_\delta(\Omega \setminus E) > \lambda_\delta(\Omega). \quad (3.40)$$

Proof. Let $V = \Omega \setminus E$. We may assume V has positive measure, for inequality (3.40) is trivial otherwise.

Suppose to the contrary that $\lambda_\delta(V) = \lambda_\delta(\Omega)$. Since V has positive measure, Proposition 3.13 provides a function $u \in A(V)$ with $Q_\delta(u) = \lambda_\delta(V)$. Hence u also is a minimizer of Q_δ over $A(\Omega)$, and Proposition 3.13 implies that u weakly solves

$$-a\Delta u + \left[R - \lambda_\delta \rho^{2(\delta^* - \delta)}\right] u = 0 \quad (3.41)$$

on Ω . Local regularity implies that $u \in W_{\text{loc}}^{2,p}(\Omega)$, and we may assume after adjusting u on a set of zero measure that u is continuous. Since E has positive measure, we can still conclude that u vanishes at some point in Ω . Following the argument of Lemma 4 from [Max05b], we may apply the weak Harnack inequality of [Tru73] to conclude that u vanishes everywhere on the connected set Ω , and hence on all of M . Since $u \in A(\Omega)$, this is a contradiction. \square

The connectivity hypothesis in Lemma 3.16 is necessary to obtain strict monotonicity. For example, two disjoint unit balls in R^n have the same first eigenvalue as a single unit ball. On the other hand, the assumption that Ω is open is not optimal, and relaxing this condition would require a suitable replacement for the weak Harnack inequality.

Although we have not established continuity from below for λ_δ , it holds in certain cases. The following is a prototypical result that suffices for our purposes.

Lemma 3.17 (Continuity from below; prototype). *Suppose V is measurable. Let $x_0 \in M$ and let $B_r(x_0)$ be the ball of radius r about x_0 . Then for any $\delta > \delta^*$*

$$\lim_{r \rightarrow 0} \lambda_\delta(V \setminus B_r) = \lambda_\delta(V). \quad (3.42)$$

Proof. Let u be a function in $A(V)$ that minimizes Q_δ . Let χ_r be a radial bump function that equals 0 on $B_r(x_0)$, equals 1 outside $B_{2r}(x_0)$, and has gradient bounded by $2/r$. Defining $u_r = \chi_r u$ we claim that $u_r \rightarrow u$ in $W_{\delta^*}^{1,2}(M)$. Assuming this for the moment, we conclude from the continuity of Q_δ that

$$\lambda_\delta(V) \leq \lambda_\delta(V \setminus B_r) \leq Q_\delta(u_r) \rightarrow Q_\delta(u) = \lambda_\delta(V) \quad (3.43)$$

and hence we obtain equality (3.42).

To show $u_r \rightarrow u$ in $W_{\delta^*}^{1,2}$, since $u_r \rightarrow u$ in $L_{\delta^*}^2$, it is enough to show that $\int |\nabla(u - u_r)|^2 \rightarrow 0$. However,

$$\int |\nabla(u - u_r)|^2 \leq 2 \int (1 - \chi_r)^2 |\nabla u|^2 + u^2 |\nabla(1 - \chi_r)|^2. \quad (3.44)$$

The first term on the right-hand side of inequality (3.44) evidently converges to zero. For the second, we note from Hölder's inequality that

$$\int_{B_{2r}} u^2 \leq \left[\int_{B_{2r}} u^N \right]^{\frac{2}{N}} \left[\int_{B_{2r}} 1 \right]^{\frac{2}{N}} \leq C r^2 \left[\int_{B_{2r}} u^N \right]^{\frac{2}{N}}. \quad (3.45)$$

Since $u \in L_{\text{loc}}^N$, $\int_{B_{2r}} u^N \rightarrow 0$ as $r \rightarrow 0$. Since $\nabla(1 - \chi_r)$ is bounded by c/r , we conclude that the second term of the right-hand side of inequality (3.44) also converges to zero. \square

4 Prescribed Non-Positive Scalar Curvature

In this section we prove the following necessary and sufficient condition for being able to conformally transform to non-positive scalar curvature for AE manifolds with at least one end.

Theorem 4.1. *Let (M^n, g) be a $W_\tau^{2,p}$ AE manifold with $p > n/2$ and $\tau \in (2 - n, 0)$. Suppose $R' \in L_{\tau-2}^p$ is non-positive. Then the following are equivalent:*

1. *There exists a positive function ϕ with $\phi - 1 \in W_\tau^{2,p}$ and such that the scalar curvature of $g' = \phi^{N-2}g$ is R' .*
2. *$\{R' = 0\}$ is Yamabe positive.*

For compact Yamabe negative manifolds we have the following analogous result. Since Rauzy's condition (1.2) is equivalent to the set $\{R' = 0\}$ being Yamabe positive, this is a generalization to lower regularity and a correction of the proof of part of [Rau95] Theorem 1.

Theorem 4.2. *Let (M^n, g) be a $W^{2,p}$ compact Yamabe negative manifold with $p > n/2$. Suppose $R' \in L^p$ is non-positive. Then the following are equivalent:*

1. *There exists a positive function ϕ with $\phi \in W^{2,p}$ and such that the scalar curvature of $g' = \phi^{N-2}g$ is R' .*
2. *$\{R' = 0\}$ is Yamabe positive.*

For the most part, the proof of Theorem 4.2 can be obtained from the proof of Theorem 4.1 by treating a compact manifold as an asymptotically Euclidean manifold with zero ends. So we focus on Theorem 4.1 and then present the few additional arguments needed for Theorem 4.2 at the end of the section.

Turning to Theorem 4.1, the proof that 1) implies 2) is short, so we delay it and concentrate on the direction 2) implies 1). Suppose that $\{R' = 0\}$ is Yamabe positive. We will show that we can make the desired conformal change via a sequence of results proved over the remainder of this section. It suffices work under the following simplifying hypotheses.

1. We may assume that the prescribed scalar curvature R' is bounded since Lemma 4.3, which we prove next, shows that we can lower scalar curvature after first solving the problem for a scalar curvature that is truncated below.
2. We may assume $\{R' = 0\}$ contains a neighborhood of infinity, since continuity from above (Lemma 3.14) shows we can truncate R' in a “small” neighborhood of infinity such that its zero set remains Yamabe positive, and we can subsequently lower scalar curvature after solving the modified problem.
3. We may assume that the initial scalar curvature satisfies $R = 0$ in a neighborhood of infinity, since Lemma 4.4, which we prove below, shows we can initially conformally transform to such a scalar curvature, and since the hypotheses of Theorem 4.1 are conformally invariant.

Lemma 4.3. *Suppose (M, g) is a $W_\tau^{2,p}$ AE manifold with $p > n/2$ and $\tau \in (2 - n, 0)$. Suppose $R' \in L_{\tau-2}^p$. If $R_g \geq R'$, then there exists a positive ϕ with $\phi - 1 \in W_\tau^{2,p}$ such that $g' = \phi^{N-2}g$ has scalar curvature R' .*

Proof. We seek a solution to $-a\Delta\phi + R\phi = R'\phi^{q-1}$. Note that 0 is a subsolution and, since $R \geq R'$, 1 is a supersolution. By [Max05b] Proposition 2, there exists a solution ϕ with $0 \leq \phi \leq 1$ and $\phi - 1 \in W_\tau^{2,p}$. Since $\phi \geq 0$ solves $-a\Delta\phi + (R - R'\phi^{q-2})\phi = 0$, and since $\phi \rightarrow 1$ at infinity, the weak Harnack inequality [Tru73] implies ϕ is positive. \square

Lemma 4.4. *Suppose (M, g) is a $W_\tau^{2,p}$ AE manifold with $p > n/2$ and $\tau \in (2 - n, 0)$. There exists $\phi > 0$ with $\phi - 1 \in W_\tau^{2,p}$ such that the metric $g' = \phi^{N-2}g$ has zero scalar curvature on some neighborhood of infinity.*

Proof. We prove the result for a manifold with one end; the extension to several ends can be done by repeated application of our argument. Let E_r be the region outside the coordinate ball of radius r in end coordinates. By Lemma 3.15, $y(E_r) > 0$ for r large enough. Following [Max05b] Proposition 3 we claim that

$$-a\Delta + \eta R : \{u \in W_\tau^{2,p}(E_r) : u|_{\partial E_r} = 0\} \rightarrow L_{\tau-2}^p(E_r) \quad (4.1)$$

is an isomorphism for all $\eta \in [0, 1]$. Because we assume homogenous boundary conditions, the argument in [Bar86] Propositions 1.6 through 1.14 showing that $-a\Delta + \eta R$ is Fredholm of index zero requires no changes except imposing the boundary condition. Suppose, then, to produce a contradiction, that there exists a nontrivial u in the kernel. An argument parallel to [Max05b] Lemma 3 implies $u \in W_{\tau'}^{2,p}$ for any $\tau' \in (2 - n, 0)$. In particular, the extension of u by zero to M belongs to $W_{\delta^*}^{1,2}(M)$ and hence also to $A(E_r)$. Integration by parts implies $Q^y(u) = 0$, which contradicts the fact that E_r is Yamabe positive. Thus $-a\Delta + \eta R$ is an isomorphism.

Let u_η be the nontrivial solution in $\{u \in W_\tau^{2,p}(E_r) : u|_{\partial E_r} = 0\}$ of

$$-a\Delta u_\eta + \eta R u_\eta = -\eta R. \quad (4.2)$$

Then $\phi_\eta := u_\eta + 1$ solves

$$-a\Delta\phi_\eta + \eta R\phi_\eta = 0 \quad (4.3)$$

on E_r . Let $I = \{\eta \in [0, 1] : \phi_\eta > 0\}$. Since $\phi_0 \equiv 1$, I is nonempty. The set of u_η such that $u_\eta > -1$ is open in $W_\tau^{2,p} \subset C_\tau^0$. Thus, by the continuity of the map $\eta \mapsto u_\eta$, I is open. Suppose $\eta_0 \in \bar{I}$. If $\phi_{\eta_0} = 0$ somewhere, the weak Harnack inequality [Tru73] implies that $\phi_{\eta_0} \equiv 0$, which contradicts the fact that $\phi_{\eta_0} \rightarrow 1$ at infinity. Thus $\phi_{\eta_0} > 0$ on E_r , and so I is closed. Thus $I = [0, 1]$, and $\phi_1 > 0$. Let ϕ be an arbitrary positive $W_\tau^{2,p}$ extension of $\phi_1|_{E_r}$. \square

Consider the family of functionals

$$F_q(u) = \int a|\nabla u|^2 + \int R(u+1)^2 - \frac{2}{q} \int R'|u+1|^q \quad (4.4)$$

for $q \in [2, N)$.

Broadly, the strategy of the proof is to construct minimizers u_q of the subcritical functionals, and then establish sufficient control to show that $(1 + u_q)$ converges in the limit $q \rightarrow N$ to the desired conformal factor. The following uniform coercivity estimate, which we prove following a variation of techniques found in [Rau95], is the key step in showing the existence of subcritical minimizers.

Proposition 4.5 (Coercivity of F_q). *Suppose $\{R' = 0\}$ is Yamabe positive, let $\delta > \delta^*$, and let $q_0 \in (2, N)$. For every $B \in \mathbb{R}$ there is a $K > 0$ such that for all $q \in [q_0, N)$ and $u \in W_{\delta^*}^{1,2}$ with $u \geq -1$, if $\|u\|_{2,\delta} > K$ then $F_q(u) > B$.*

Proof. For $\eta > 0$ let

$$A_\eta = \left\{ u \in W_{\delta^*}^{1,2}, u \geq -1 : \int |R'| |u|^2 \leq \eta \|u\|_{2,\delta}^2 \int |R'| \right\}. \quad (4.5)$$

Morally, $u \in A_\eta$ if it is concentrated on the zero set

$$Z = \{R' = 0\}, \quad (4.6)$$

with greater concentration as $\eta \rightarrow 0$.

Fix $\mathcal{L} \in (0, \lambda_\delta(Z))$. We first claim that there is an $\eta_0 < 1$ such that if $u \in A_{\eta_0}$, then

$$\int a |\nabla u|^2 + R u^2 \geq \mathcal{L} \|u\|_{2,\delta}^2. \quad (4.7)$$

Suppose to the contrary that this is false, and let η_k be a sequence converging to 0. We can then construct a sequence v_k with each $v_k \in A_{\eta_k}$ such that $\|v_k\|_{2,\delta} = 1$ and

$$\int a |\nabla v_k|^2 + R v_k^2 < \mathcal{L}. \quad (4.8)$$

Note that \mathcal{L} is finite even if $\lambda_\delta(Z) = \infty$. So from the boundedness of the sequence v_k in L_δ^2 and Lemma 3.5, the sequence is bounded in $W_{\delta^*}^{1,2}$, and a subsequence (which we reduce to) converges weakly in $W_{\delta^*}^{1,2}$ and strongly in L_δ^2 to a limit v with $\|v\|_{2,\delta} = 1$. Now

$$0 \leq \int |R'| v_k^2 \leq \eta_k \int |R'| \rightarrow 0. \quad (4.9)$$

Since $|R'| v_k^2 \rightarrow |R'| v^2$ in L^1 we conclude $v = 0$ outside of Z . From weak upper semicontinuity (Corollary 3.2) we conclude

$$\int a |\nabla v|^2 + R v^2 \leq \mathcal{L} \quad (4.10)$$

as well. However, since v is supported in Z

$$\int a |\nabla v|^2 + R v^2 \geq \lambda_\delta(Z) \|v\|_{2,\delta}^2 = \lambda_\delta(Z) > \mathcal{L}, \quad (4.11)$$

which is a contradiction, and establishes inequality (4.7).

Let $B \in \mathbb{R}$ and suppose $q \in (q_0, N)$, $u \in W_{\delta^*}^{1,2}$ and $u \geq -1$. We wish to show that there is a K independent of q so that if $\|u\|_{2,\delta} > K$ then $F_q(u) > B$. It is enough to find a choice of K under two cases depending on whether $u \in A_{\eta_0}$ or not. When u is concentrated on Z , the coercivity will follow from the fact that Z is Yamabe positive (as used to obtain inequality (4.7)), and when u is not concentrated on Z the coercivity will follow from the fact that $R' < 0$ away from Z .

Suppose that $u \notin A_{\eta_0}$, so

$$\int |R'| |u|^2 > \eta_0 \|u\|_{2,\delta}^2 \int |R'|. \quad (4.12)$$

We calculate

$$\begin{aligned}
F_q(u) &= \int a|\nabla u|^2 + \int R(u+1)^2 + \frac{2}{q} \int |R'| |u+1|^q \\
&\geq \int a|\nabla u|^2 - 2 \int |R|(u^2+1) + \frac{2}{q} \int |R'|(|u|^q - 1) \\
&\geq \int \frac{a}{2} |\nabla u|^2 - C \|u\|_{2,\delta}^2 - 2 \int |R| + \frac{2}{q} \int |R'|(|u|^q - 1) \\
&\geq \int \frac{a}{2} |\nabla u|^2 - C \|u\|_{2,\delta}^2 - 2 \int \left(|R| + \frac{1}{q} |R'| \right) + \frac{2}{q} \int |R'| |u|^q.
\end{aligned} \tag{4.13}$$

Here we have applied Lemma 3.1 to determine the constant $C > 0$, and have used the fact that $(u+1)^q \geq |u|^q - 1$ for $u \geq -1$. Inequality (4.12) and Hölder's inequality imply

$$\eta_0 \|u\|_{2,\delta}^2 \int |R'| < \int |R'| |u|^2 \leq \left(\int |R'| |u|^q \right)^{\frac{2}{q}} \left(\int |R'| \right)^{1-\frac{2}{q}} \tag{4.14}$$

and hence

$$(\eta_0)^{\frac{q}{2}} \|u\|_{2,\delta}^q \int |R'| \leq \int |R'| |u|^q. \tag{4.15}$$

Using the fact that $\eta_0 < 1$ and $q < N$, inequalities (4.13) and (4.15) imply at last that

$$F_q(u) \geq \int \frac{a}{2} |\nabla u|^2 - C \|u\|_{2,\delta}^2 - 2 \int \left(|R| + \frac{1}{q} |R'| \right) + \frac{2}{q} (\eta_0)^{\frac{N}{2}} \|u\|_{2,\delta}^q \int |R'|. \tag{4.16}$$

We note that $\int |R'| > 0$, for otherwise condition (4.12) is impossible, and hence the coefficient on $\|u\|_{2,\delta}^q$ is positive. Since $q > 2$, there is a K such that if $\|u\|_{2,\delta} > K$, $F_q(u) \geq B$. Note that since C is independent of $q \geq q_0$, so is the choice of K .

Now suppose $u \in A_{\eta_0}$, so inequality (4.7) holds. Then for any $\epsilon > 0$,

$$\begin{aligned}
F_q(u) &\geq \int a|\nabla u|^2 + \int R(u+1)^2 \\
&= \int a|\nabla u|^2 + Ru^2 + \int R[(u+1)^2 - u^2] \\
&\geq \int a|\nabla u|^2 + Ru^2 - \int |R| \left[\epsilon u^2 + 1 + \frac{1}{\epsilon} \right] \\
&\geq (1-\epsilon) \left[\int a|\nabla u|^2 + Ru^2 \right] + \epsilon \int (a|\nabla u|^2 - 2|R|u^2) - \left(1 + \frac{1}{\epsilon} \right) \int |R| \\
&\geq (1-\epsilon) \mathcal{L} \|u\|_{2,\delta}^2 + \epsilon \left(\int \frac{a}{2} |\nabla u|^2 - C \|u\|_{2,\delta}^2 \right) - \left(1 + \frac{1}{\epsilon} \right) \int |R| \\
&\geq [(1-\epsilon)\mathcal{L} - \epsilon C] \|u\|_{2,\delta}^2 + \epsilon \int \frac{a}{2} |\nabla u|^2 - \left(1 + \frac{1}{\epsilon} \right) \int |R|.
\end{aligned} \tag{4.17}$$

Here we have applied Lemma 3.1 to determine the constant C , inequality (4.7), and the fact that $(u+1)^2 - u^2 \leq \epsilon u^2 + 1 + (1/\epsilon)$ for all $u \geq -1$ and all $\epsilon > 0$. We can pick ϵ sufficiently small such that the coefficient of $\|u\|_{2,\delta}^2$ in the final expression of inequality (4.17) is at least $\mathcal{L}/2$. Hence there is a K such that if $\|u\|_{2,\delta} \geq K$, $F_q(u) \geq B$. Since C is independent of $q \geq q_0$, so is ϵ and the choice of K . \square

Lemma 4.6. *For $q < N$ the operator F_q is weakly upper semicontinuous on $W_{\delta^*}^{1,2}$.*

Proof. Lemma 3.1 together with the weak continuity of continuous linear maps implies

$$u \mapsto \int a|\nabla u|^2 + R(u+1)^2 \quad (4.18)$$

is weakly upper semicontinuous on $W_{\delta^*}^{1,2}$. Hence it suffices to show that

$$u \mapsto \int R'|u+1|^{q-1} \quad (4.19)$$

is weakly continuous on $W_{\delta^*}^{1,2}$. But fixing $\delta > \delta^*$ we know that the embedding $W_{\delta^*}^{1,2} \hookrightarrow L_{\delta}^q$ is compact and that the map (4.19) is continuous on L_{δ}^q . \square

We now obtain existence of subcritical minimizers from the coercivity of F_q , along with uniform estimates in $W_{\delta^*}^{1,2}$ for the minimizers.

Lemma 4.7. *For any $q_0 \in (2, N)$, for each $q \in [q_0, N)$, there exists $u_q > -1$, bounded in $W_{\delta^*}^{1,2}$ independent of q , which minimizes F_q and is a weak solution of*

$$-a\Delta(u_q+1) + R(u_q+1) = R'(u_q+1)^{q-1}. \quad (4.20)$$

Moreover, $u_q \in W_{\sigma}^{2,p}$ for every $\sigma \in (2-n, 0)$.

Proof. Let $B = \int R + \int |R'|$, let $\delta > \delta^*$, and let $q_0 \in (2, N)$. Observe that

$$F_q(0) \leq B \quad (4.21)$$

for all $q \in (q_0, N)$. Let K be the constant associated with B , δ and q_0 obtained from Proposition 4.5. Fix $q \in (q_0, N)$ and let u_k be a minimizing sequence in $W_{\delta^*}^{1,2}$ for F_q . Without loss of generality, we can assume each $u_k \geq -1$ since $F_q(u_k) = F_q(\max(u_k, -2-u_k))$. We can assume that each $F_q(u_k) \leq F_q(0) \leq B$ and hence Proposition 4.5 implies each $\|u_k\|_{2,\delta} \leq K$. Since

$$\int a|\nabla u_k|^2 + R(1+u_k)^2 \leq F_q(u_k) < B \quad (4.22)$$

as well, Lemma 3.5 implies that there is a $C > 0$ such that each $\|u_k\|_{W_{\delta^*}^{1,2}} \leq C$. Note that C depends on K and B , which are independent of $q \geq q_0$. A subsequence (which we reduce to) converges weakly in $W_{\delta^*}^{1,2}$ and strongly in L_{δ}^q to a limit $u_q \geq -1$. Lemma 4.6 shows F_q is weakly upper semicontinuous, so u_q is a minimizer. Moreover, $\|u_q\|_{W_{\delta^*}^{1,2}} \leq C$ as well.

Since u_q is a minimizer, we find that $(1+u_q)$ is a weak solution of

$$\left[-a\Delta + R - R'(1+u_q)^{q-2}\right](1+u_q) = 0. \quad (4.23)$$

Since $R' \in L_{\text{loc}}^{\infty}$ and since $u_q \in L_{\text{loc}}^N$, an easy computation shows that $R'(1+u_q)^{q-2} \in L_{\text{loc}}^r$ for some $r > n/2$. Since $R \in L_{\text{loc}}^p$ and $g \in W_{\text{loc}}^{2,p}$ with $p > n/2$, we find that the coefficients of the differential operator in brackets

in equation (4.23) satisfy the hypotheses of the weak Harnack inequality of [Tru73]. Hence, since $1 + u_q \geq 0$ and since the manifold is connected, either $1 + u_q > 0$ everywhere or $u_q \equiv -1$. But u_q decays at infinity, and so we conclude that $1 + u_q$ is everywhere positive.

We now bootstrap the regularity of u_q , which we know initially belongs to $L_{\delta^*}^N$. Fix $\sigma \in (2 - n, 0)$. Suppose it is known that for some $r \geq N$ that $u_q \in L_{\text{loc}}^r$. From equation (4.23), u_q solves

$$-a\Delta u_q = R'(1 + u_q)^{q-1} - R(1 + u_q). \quad (4.24)$$

Recall that $R' \in L_{\text{loc}}^\infty$ and $R \in L_{\text{loc}}^p$ and both have compact support. Then $R'(1 + u_q)^{q-1}$ belongs to $L_\sigma^{t_1}$ with

$$\frac{1}{t_1} = \frac{q-1}{r} \leq \frac{1}{r} + \frac{q-2}{N} < \frac{N-1}{N} \quad (4.25)$$

and $R(1 + u_q)$ belongs to $L_\sigma^{t_2}$ with

$$\frac{1}{t_2} = \frac{1}{r} + \frac{1}{p}. \quad (4.26)$$

Let $t = \min(t_1, t_2)$ and note that $t < p$ since $t_2 < p$. From [Bar86] Proposition 1.6 we see that u_q is a strong solution of (4.24) and from [Bar86] Proposition 2.2, which implies $\Delta : W_\sigma^{2,t} \rightarrow L_\sigma^t$ is an isomorphism for $1 < t \leq p$, we conclude that $u_q \in W_\sigma^{2,t}$. From Sobolev embedding we obtain $u_q \in L_\sigma^{r'}$ where

$$\frac{1}{r'} = \frac{1}{t} - \frac{2}{n}, \quad (4.27)$$

so long as $1/t > n/2$, at which point the bootstrap changes as discussed below. Now

$$\begin{aligned} \frac{1}{t_1} - \frac{2}{n} &\leq \frac{1}{r} + \frac{q-2}{N} - \frac{2}{n} \\ &= \frac{1}{r} + \frac{q}{N} - \left[\frac{2}{N} + \frac{2}{n} \right] \\ &= \frac{1}{r} + \left[\frac{q}{N} - 1 \right]. \end{aligned} \quad (4.28)$$

Also,

$$\frac{1}{t_2} - \frac{2}{n} = \frac{1}{r} + \left[\frac{1}{p} - \frac{2}{n} \right]. \quad (4.29)$$

Let $\epsilon = \min(1 - q/N, 2/n - 1/p)$ and note that ϵ is positive and independent of r . Inequalities (4.28) and (4.29) imply

$$\frac{1}{r'} \leq \frac{1}{r} - \epsilon \quad (4.30)$$

Hence, after a finite number of iterations (depending on the size of ϵ , and hence on how close q is to N) we can reduce $1/r$ by multiples of ϵ until $1/r \leq \epsilon$. At this point the bootstrap changes, and in at most two more iterations we can conclude that $u_q \in L_\sigma^\infty$ and also $u_q \in W_\sigma^{2,p}$.

□

The uniform $W_{\delta^*}^{1,2}$ bounds of Lemma 4.7 are enough to obtain the existence of a solution u in $W_\sigma^{2,N/(N-1)}$ of equation (4.20) with $q = N$. At the end of Section IV.6 of [Rau95] it is claimed that on a compact manifold

in the smooth setting that elliptic regularity now implies u is smooth. But in fact this is not quite enough regularity to start a bootstrap: $W_\sigma^{2,N/(N-1)}$ embeds continuously in L_σ^N , which is no more regularity than was known initially. To start a bootstrap and ensure the continuity of u we need the following improved estimate, which follows a modification of the strategy of [LP87] Proposition 4.4.

Lemma 4.8. *For each compact set K , the minimizers u_q are uniformly bounded in $L^M(K)$ for some $M > N$.*

Proof. Let χ be a smooth positive function with compact support that equals 1 in a neighborhood of K . Let $v = \chi^2(1 + u_q)^{1+2\sigma}$ where u_q is a subcritical minimizer and where σ is a small constant to be chosen later. Note that since $u_q \in L_{\text{loc}}^\infty \cap W_{\text{loc}}^{1,2}$, $v \in W_{\delta^*}^{1,2}$. Setting $w = (1 + u_q)^{1+\sigma}$, a short computation shows that

$$\int \chi^2 |\nabla w|^2 = -2 \frac{1+\sigma}{1+2\sigma} \int \langle \chi \nabla w, w \nabla \chi \rangle + \frac{(1+\sigma)^2}{1+2\sigma} \int \langle \nabla u_q, \nabla v \rangle. \quad (4.31)$$

Applying Young's inequality to the first term on the right-hand side of equation (4.31) and merging a resulting piece into the left-hand side we conclude there is a constant C_1 such that

$$\|\chi \nabla w\|_2^2 \leq C_1 \|w \nabla \chi\|_2^2 + 2 \frac{(1+\sigma)^2}{1+2\sigma} \int \langle \nabla u_q, \nabla v \rangle. \quad (4.32)$$

Since u_q is a subcritical minimizer,

$$\begin{aligned} a \int \langle \nabla u_q, \nabla v \rangle &= \int R'(1 + u_q)^{q-2} \chi^2 w^2 - \int R \chi^2 w^2 \\ &\leq \left| \int R \chi^2 w^2 \right| \\ &\leq \epsilon \|\nabla(\chi w)\|_2^2 + C_\epsilon \|\chi w\|_2^2. \end{aligned} \quad (4.33)$$

We applied Lemma 3.1 in the last line and used the fact that for functions with support contained in a fixed compact set, weighted and unweighted norms are equivalent. Note also that obtaining line 2 used the fact that $R' \leq 0$ everywhere. Noting that there is a constant C_2 such that

$$\|\nabla(\chi w)\|_2^2 \leq C_2 (\|\chi \nabla w\|_2^2 + \|w \nabla \chi\|_2^2), \quad (4.34)$$

we can combine inequalities (4.32), (4.33), and (4.34) to conclude that, upon taking ϵ sufficiently small to absorb the term from inequality (4.33) into the left-hand side, there is a constant C_3 such that

$$\|\nabla(\chi w)\|_2^2 \leq C_3 [\|w \nabla \chi\|_2^2 + \|w \chi\|_2^2]. \quad (4.35)$$

Finally, from the Sobolev inequality (2.11), there is a constant C_4 such that

$$\|\chi w\|_N^2 \leq C_4 [\|w \nabla \chi\|_2^2 + \|w \chi\|_2^2] \quad (4.36)$$

as well. Now u_q is bounded uniformly in L^N on the support K' of χ , and hence we can take σ sufficiently small so that w is bounded independent of q in $L^2(K')$ as well. Thus $(1 + u_q)$ is bounded uniformly in $L^M(K)$ for $M = N(1 + \sigma)$. \square

Corollary 4.9. *Let p be the exponent such that g is a $W_\tau^{2,p}$ AE manifold and let $\sigma \in (2-n, 0)$. The subcritical minimizers u_q are bounded in $W_\sigma^{2,p}$ as $q \rightarrow N$.*

Proof. Consider a subcritical minimizer u_q , which is a weak solution of

$$-a \Delta u_q = -R(1 + u_q) + R'(1 + u_q)^{q-1}. \quad (4.37)$$

Let K be a compact set containing the support of R and R' , and let $M > N$ be an exponent such that we have uniform bounds on u_q in $L^M(K)$. We wish to bootstrap this to better regularity for u_q .

Since the bootstrap for the two terms is different, we concentrate first on the interesting term, $R'(1 + u_q)^{q-1}$, and suppose for the moment that the other term is absent. Let us write

$$\frac{1}{M} = \frac{1}{N} - \epsilon \quad (4.38)$$

for some $\epsilon > 0$. Now

$$|R'(1 + u_q)^{q-1}| \leq |R'|(1 + |1 + u_q|^{N-1}). \quad (4.39)$$

Since R' is bounded, the term $R'|1 + u_q|^{N-1}$ belongs to $L^s(K)$ with

$$\begin{aligned} \frac{1}{s} &= \frac{1}{M}(N-1) \\ &= \left(\frac{1}{N} - \epsilon\right)(N-1) \\ &= \frac{2}{n} + \frac{1}{N} - \epsilon(N-1). \end{aligned} \quad (4.40)$$

Since R' is zero outside of K we conclude $R'(1 + u_q)^{q-1} \in L_{\sigma}^s$. Note that the norm of $R'(1 + u_q)^{q-1}$ in L_{σ}^s depends on the norm of u_q in $L^M(K)$ but is otherwise independent of q . Since the functions u_q are uniformly bounded in $L^M(K)$, we obtain control of $R'(1 + u_q)^{q-1}$ in L_{σ}^s independent of q .

If $s \leq p$ then $s \in (1, p]$ and we cite [Bar86] Proposition 2.2 to conclude $u_q \in W_{\sigma}^{2,s}$ and therefore $u_q \in L^{M'}(K)$ with

$$\frac{1}{M'} = \frac{1}{s} - \frac{2}{n} = \frac{1}{N} - \epsilon(N-1). \quad (4.41)$$

Similarly, after k iterations of this process we would find u_q belongs to $W_{\sigma}^{2,s}$ with

$$\frac{1}{s} = \frac{2}{n} + \frac{1}{N} - \epsilon(N-1)^k \quad (4.42)$$

unless $s > p$, at which point the bootstrap terminates at $u_q \in W_{\sigma}^{2,p}$ with norm depending on $\|u_q\|_{L^M(K)}$ (which is independent of q) and the number of iterations needed to reach $s \leq p$. Note that since $N > 2$, we will reach the condition $s \geq p$ in a finite number of steps independent of q .

Now consider the bootstrap for the term $-R(1 + u_q)$ alone. Write

$$\frac{1}{p} = \frac{2}{n} - \epsilon' \quad (4.43)$$

for some $\epsilon' > 0$. The term $-R(1 + u_q)$ then belongs to $L^t(K)$ with

$$\frac{1}{t} = \frac{1}{p} + \frac{1}{M} = \frac{2}{n} - \epsilon' + \frac{1}{M}. \quad (4.44)$$

Note that $1 < t < p$ and hence [Bar86] Proposition 2.2 implies $u_q \in W_{\sigma}^{2,t}$. Note that the norm of u_q in $W_{\sigma}^{2,t}$ depends on the norm of u_q in $L^M(K)$ but is otherwise independent of q . Consequently u_q is controlled in $L^{M'}(K)$ independent of q where

$$\frac{1}{M'} = \frac{1}{t} - \frac{2}{n} = \frac{1}{M} - \epsilon'. \quad (4.45)$$

After k iterations we would find instead

$$\frac{1}{M'} = \frac{1}{M} - k\epsilon' \quad (4.46)$$

and the bootstrap stops in finitely many steps independent of q when $k\epsilon' > 1/M$, at which point we find that $u_q \in W_{\sigma}^{2,p}$, with norm independent of q . There is an exceptional case if $k\epsilon' = 1/M$, but it can be avoided by an initial perturbation of M .

The bootstrap in the full case follows from combining these arguments. \square

Proof of Theorem 4.1. (2. implies 1.) The u_q are uniformly bounded in $W_{\sigma}^{2,p}$ by Corollary 4.9 for any $\sigma \in (2-n, 0)$. Thus they converge to some u strongly in $W_{\delta^*}^{1,2}$ and uniformly on compact sets. In particular, since the u_q weakly solve (4.20), $\phi := u + 1$ weakly solves

$$-a\Delta\phi + R\phi = R'\phi^{N-1}. \quad (4.47)$$

Since each $u_q \geq -1$, $\phi \geq 0$, and since $\phi \rightarrow 1$ at infinity, $\phi \not\equiv 0$. Hence the weak Harnack inequality [Tru73] implies $\phi > 0$.

Since $\sigma \in (2-n, 0)$ is arbitrary, $\phi - 1 \in W_{\tau}^{2,p}$ in particular. Note that the rapid decay $\sigma \approx 2-n$ uses the fact that $R = 0$ near infinity. The lesser decay rate τ in the statement of the theorem stems from the fact that we may have used a conformal factor in $W_{\tau}^{2,p}$ to initially set $R = 0$ near infinity or to lower the scalar curvature after changing it to R' .

(1. implies 2.) Let $Z = \{R' = 0\}$. The case where Z has zero measure is trivial, for then $y(Z) = \infty > 0$. Hence we assume Z has positive measure and suppose there exists a conformally related metric g' with scalar curvature R' . Let $\delta > \delta^*$ be fixed and let u be a minimizer of $Q_{g',\delta}$ over $A(Z)$ as provided by Proposition 3.13. Note that

$$\int R' u^2 dV_{g'} = 0 \quad (4.48)$$

since $R' = 0$ on Z and $u = 0$ on Z^c . Hence

$$\lambda_{g',\delta}(Z) = Q_{g',\delta}(u) = a \frac{\int |\nabla u|_{g'}^2 dV_{g'}}{\|u\|_{g',2,\delta}^2}. \quad (4.49)$$

In particular, $\lambda_{g',\delta}(Z) \geq 0$, and $\lambda_{g',\delta}(Z) = 0$ only if u is constant. But Z has positive measure, and therefore $A(Z)$ does not contain any constants. Hence $\lambda_{g',\delta}(Z) > 0$, and Proposition 3.10 implies Z is Yamabe positive. \square

This completes the proof of Theorem 4.1. Turning to the compact case (Theorem 4.2) recall that we started the AE argument with the following inessential simplifying hypotheses:

1. The prescribed scalar curvature R' is bounded.

2. The prescribed scalar curvature R' has compact support.
3. The initial scalar curvature R has compact support.

The last two of these are trivial when M is compact, and the first is justified by Lemma 4.10 below, which shows that we can lower scalar curvature after first solving the problem for a scalar curvature that is truncated below. In the compact case we require an additional inessential condition which will be used in Lemma 4.11.

4. We may assume that the initial scalar curvature R is continuous and negative. Indeed, from Proposition 3.13 there is a positive function ϕ solving $-a\Delta\phi + R\phi = \lambda_\delta(M)\phi$ on M . Note that $\lambda_\delta(M) < 0$ since g is Yamabe negative. Using ϕ as the conformal factor we obtain a scalar curvature $\lambda_\delta(M)\phi^{2-N}$. The hypotheses of Theorem 4.2 are conformally invariant and hence unaffected by this change.

Lemma 4.10. *Suppose (M, g) is a $W^{2,p}$ compact Yamabe negative manifold. Suppose $R' \in L^p$. If $0 \geq R \geq R'$, then there exists a positive ϕ with $\phi \in W^{2,p}$ such that $g' = \phi^{N-2}g$ has scalar curvature R' .*

Proof. We wish to solve

$$-a\Delta\phi + R\phi = R'\phi^{N-1}. \quad (4.50)$$

Note that $\phi_+ = 1$ is a supersolution of equation (4.50). To find a subsolution first observe that $R \neq 0$ since the manifold is Yamabe negative. So, since $-R \geq 0$ and $-R \neq 0$, for each $\epsilon > 0$ there exists a unique $\phi_\epsilon \in W^{2,p}$ solving

$$-a\Delta\phi_\epsilon - R\phi_\epsilon = -R + \epsilon R'. \quad (4.51)$$

When $\epsilon = 0$ the solution is 1, and since $W^{2,p}$ embeds continuously in C^0 we can fix $\epsilon > 0$ such that $\phi_\epsilon > 1/2$ everywhere. We claim that $\phi_- := \eta\phi_\epsilon$ is a subsolution if $\eta > 0$ is sufficiently small. Indeed,

$$\begin{aligned} -a\Delta\phi_- + R\phi_- &= \eta[R(2\phi_\epsilon - 1)] + \eta\epsilon R' \\ &\leq \eta\epsilon R'. \end{aligned} \quad (4.52)$$

So ϕ_- is a subsolution so long as

$$\eta\epsilon R' \leq R'\phi_-^{N-1} \quad (4.53)$$

A quick computation shows that inequality (4.53) holds if η is small enough so that $\eta^{2-N} \geq \phi_\epsilon^{N-1}/\epsilon$ everywhere. We can also take η small enough so that $\phi_- \leq \phi_+ = 1$, and hence there exists a solution $\phi \in W^{2,p}$ with $\phi \geq \phi_- > 0$ of equation (4.50) ([Max05b] Proposition 2). \square

The remainder of the proof of Theorem 4.2 nearly exactly follows the proof of Theorem 4.1 by treating a compact manifold as an asymptotically Euclidean manifold with zero ends. In particular, the cited results of Section 3 apply equally in both cases, and differences arise only when the following facts are cited.

- A constant function in $W_{\delta^*}^{1,2}$ is identically zero.
- The Laplacian is an isomorphism from $W_{\sigma}^{2,p}$ to L_{σ}^p for $\sigma \in (2-n, 0)$.

We use the property that constants in $W_{\delta^*}^{1,2}$ vanish just twice: once in Lemma 4.7 in showing $1 + u_q \neq 0$, and once in the final proof of Theorem 4.1 showing that in the limit $1 + u \neq 0$ as well. The following lemma provides the alternative argument needed to ensure these functions do not vanish identically in the compact case.

Lemma 4.11. *Suppose (M, g) is compact and that R_g is continuous and negative. Fix $q_0 \in (2, N)$. Then $\|1 + u_q\|_2 \geq C$ for some C independent of $q \in (q_0, N)$. Moreover, the limit $1 + u$ is not identically zero.*

Proof. Note that for any constant k ,

$$F_q(k) = (1 + k)^2 \int R - \frac{2}{q}(1 + k)^q \int R'. \quad (4.54)$$

Since $\int R < 0$, for any $k \neq -1$ close enough to -1 , $F_q(k) < 0$. Indeed, there are constants $k_0 > -1$ and $c > 0$ such that $F_q(k_0) < -c$ for all $q \in (q_0, N)$. But then

$$\int R(1 + u_q)^2 \leq F_q(u_q) \leq F_q(k_0) \leq -c \quad (4.55)$$

since u_q minimizes F_q . Since R is continuous, and thus bounded below, $\|1 + u_q\|_2 \geq C$ for some C independent of $q \in (q_0, N)$. Since $u_q \rightarrow u$ in L^2 , we also have $\|1 + u\|_2 \geq C$, and so $1 + u$ is not identically zero. \square

We use the fact that $\|\Delta u\|_{p,\sigma}$ controls $\|u\|_{W_{\sigma}^{2,p}}$ just twice as well, once in the bootstrap of Lemma 4.7 and once in the bootstrap of Lemma 4.9. However, on a compact manifold, $\|u\|_{W^{2,p}}$ is controlled by the sum of $\|\Delta u\|_p$ and $\|u\|_2$, and the coercivity estimate from Proposition (4.5) ensures that $\|u_q\|_2$ is uniformly bounded as $q \rightarrow N$. This provides the needed extra control for the bootstraps and completes the proof of Theorem 4.2.

5 Yamabe Classification

In this section we provide two characterizations of the Yamabe class of an asymptotically Euclidean manifold, one in terms of the prescribed scalar curvature problem and one in terms of the Yamabe type of the manifold's compactification. Note that throughout this section AE manifolds have at least one end.

Theorem 5.1. *Suppose (M, g) is a $W_{\tau}^{2,p}$ AE manifold with $p > n/2$ and $\tau \in (2 - n, 0)$. Let $\mathcal{R}_{\leq 0}$ be the set of non-positive elements of $L_{\tau-2}^p$.*

1. *M is Yamabe positive if and only if the set of non-positive scalar curvatures of metrics conformally equivalent to g is $\mathcal{R}_{\leq 0}$.*
2. *M is Yamabe null if and only if the set of non-positive scalar curvatures of metrics conformally equivalent to g is $\mathcal{R}_{\leq 0} \setminus \{0\}$.*
3. *M is Yamabe negative if and only if the set of non-positive scalar curvatures of metrics conformally equivalent to g is a strict subset of $\mathcal{R}_{\leq 0} \setminus \{0\}$.*

Proof. It suffices to prove the forward implications.

1) Suppose M is Yamabe positive, and hence so is every subset. If $R' \in \mathcal{R}_{\leq 0}$, then $\{R' = 0\}$ is Yamabe positive and Theorem 4.1. implies $[g]$ includes a metric with scalar curvature R' .

- 2) Suppose M is Yamabe null. Since M is open and connected, Lemma 3.16 implies that if $E \subseteq M$ has positive measure, then $M \setminus E$ is Yamabe positive. Hence for any $R' \in \mathcal{R}_{\leq 0}$ with $R' < 0$ on a set of positive measure, $\{R' = 0\}$ is Yamabe positive, and Theorem 4.1 implies we can conformally transform to a metric with scalar curvature R' . But $R' \equiv 0$ is impossible, for otherwise Theorem 4.1 would imply M is Yamabe positive.
- 3) Suppose M is Yamabe negative. Since M is open, Lemma 3.17 shows that there is a nonempty open set $W \subseteq M$ such that $M \setminus W$ is also Yamabe negative. Suppose $R' \in L^p_{\tau-2}$ is non-positive and supported in W . Then $\{R' = 0\}$ contains $M \setminus W$ and is hence Yamabe negative. But then Theorem 4.1 shows that we cannot conformally transform to a metric with scalar curvature R' . In particular, $R' \equiv 0$ is one of the unattainable scalar curvatures. \square

While Theorem 5.1 completely describes the set of allowable scalar curvatures in cases 1) and 2), it does not in case 3). Of course, we already have demonstrated a necessary and sufficient criterion for being able to make the conformal change: the zero set of R' must be Yamabe positive. Nevertheless, it would be desirable to describe this situation more concretely, and there are a few things that can be said. First, by Lemma 3.15, if $R' \in \mathcal{R}_{\leq 0}$ and the weighted volume of $\{R' = 0\}$ is sufficiently small, then $\{R' = 0\}$ is Yamabe positive, and thus g is conformally equivalent to a metric with scalar curvature R' . In particular, if $R' < 0$ everywhere, then it is attainable. Conversely, by Lemma 3.17, for any sequence $\{R'_k\} \subset \mathcal{R}_{\leq 0}$ such that $\{R'_k < 0\} \subset B_{1/k}(x_0)$ for some fixed $x_0 \in M$, then for k large enough, $\{R'_k = 0\}$ is Yamabe negative, and thus g is not conformally equivalent to a metric with scalar curvature R'_k . That is, the strictly negative part of R' cannot be constrained to a small ball. Similarly, an argument analogous to the proof of Lemma 3.17 shows that the complement of a sufficiently “small” neighborhood of infinity is Yamabe negative, and hence the strictly negative part of R' cannot be constrained to a small neighborhood of infinity.

Our second characterization of the Yamabe class of an AE manifold involves its compactification. An AE manifold can be compactified using a conformal factor that decays suitably at infinity, and a compact manifold can be transformed into an AE manifold using a conformal factor with a suitably singularity. We would like to show that the sign of the Yamabe invariant is preserved under these operations, and we begin by laying out the details of the compactification/decompactification procedure. In particular, there is a precise relationship between the decay of the metric at infinity and its smoothness at the point of compactification.

Lemma 5.2. *Let $p > n/2$ and let $\tau = \frac{n}{p} - 2$, so $-2 < \tau < 0$. Suppose (M, g) is a $W^{2,p}_{\tau}$ AE manifold. There is a smooth conformal factor ϕ that decays to infinity at the rate ρ^{2-n} such that $\bar{g} = \phi^{N-2}g$ extends to a $W^{2,p}$ metric on the compactification \bar{M} .*

Conversely, suppose (\bar{M}, \bar{g}) is a compact $W^{2,p}$ manifold, with $p > n/2$ and $p \neq n$. Given a finite set \mathcal{P} of points in \bar{M} there is conformal factor $\bar{\phi}$ that is smooth on $M = \bar{M} \setminus \mathcal{P}$, has a singularity of order $|x|^{2-n}$ at each point of \mathcal{P} , and such that $g = \bar{\phi}^{N-2}\bar{g}$ is a $W^{2,p}_{\tau}$ AE manifold with $\tau = \frac{n}{p} - 2$.

Proof. For simplicity we treat the case of only one end.

Let (M, g) be a $W^{2,p}_{\tau}$ AE manifold and let z^i be the Euclidean end coordinates on M , so

$$g_{ij} = e_{ij} + k_{ij}, \quad (5.1)$$

with $k \in W^{2,p}_{\tau}$. Let x^i be coordinates given by the Kelvin transform $x^i = z^i/|z|^2$, so $z^i = x^i/|x|^2$ as well.

We define a conformal factor $\phi = |z|^{2-n}$ near infinity, and extend it to be smooth on the rest of M . Let $\bar{g} = \phi^{N-2}g$ and let \bar{M} be the one-point compactification of M , with P being the point at infinity. We wish to show that \bar{g} extends to a $W^{2,p}(\bar{M})$ metric.

Near P , $\phi^{N-2} = |z|^{-4}$ and

$$\bar{g}_{ij} = e_{ij} + \bar{k}_{ij} \quad (5.2)$$

where

$$\bar{k}_{ij} := k_{ij} - \frac{4}{|x|^2} x^a k_{a(i} x_{j)} + \frac{4}{|x|^4} x^a x^b k_{ab} x_i x_j = O(k). \quad (5.3)$$

and $x_a = e_{ab} x^b$. Since $\bar{k}_{ij} \rightarrow 0$ at P , we set $\bar{g}_{ij}(P) = e_{ij}$ to obtain a continuous metric, and we need to show that $\bar{k} \in W^{2,p}(\bar{M})$. Since $\bar{k} \in W^{2,p}_{\text{loc}}(M)$, and since a point is a removable set, we need only show that the second derivatives of \bar{k} belong to $L^p(B)$ for some coordinate ball B containing P .

Let $\bar{\partial}$ represent the derivatives in x^i coordinates. Since $\frac{\partial z}{\partial x} = O(|x|^{-2})$, we calculate

$$\begin{aligned} \bar{\partial} \bar{k} &= O(\partial k) O(|z|^2) + O(k) O(|z|) \\ \bar{\partial}^2 \bar{k} &= O(\partial^2 k) O(|z|^4) + O(\partial k) O(|z|^3) + O(k) O(|z|^2). \end{aligned} \quad (5.4)$$

In order to show $\bar{\partial}^2 \bar{k} \in L^p(B)$, it is sufficient to show that each of the three terms in equation (5.4) is in $L^p(B)$.

Note that near infinity

$$d\bar{V} = \phi^N dV = |z|^{-2n} dV. \quad (5.5)$$

Hence the L^p norm of the $O(k)O(|z|^2)$ term of equation (5.4) is controlled by

$$\begin{aligned} \int (O(k)O(|z|^2))^p |z|^{-2n} dV &= \int O(|k|^p) O(|z|^{2p-2n}) dV \\ &\leq C \|k\|_{W^{2,p}_\tau}^p, \end{aligned} \quad (5.6)$$

where we have used the equality

$$2p - 2n = -n - \tau p \quad (5.7)$$

and equation (2.1) defining the weighted norm. Hence the $O(k)O(|z|^2)$ term of equation (5.4) belongs to $L^p(B)$. The two remaining terms have the same asymptotics and similar calculations show that they belong to $L^p(B)$ as well.

For the converse, consider a $W^{2,p}$ compact manifold (\bar{M}, \bar{g}) with $p > n/2$ and $p \neq n$. Let P be a point to remove to obtain $M = \bar{M} \setminus \{P\}$. Since \bar{g} is continuous we can find smooth coordinates x^i near P such that $\bar{g} = e + \bar{k}$ for some $\bar{k} \in W^{2,p}$ which vanishes at P . Moreover, if $p > n$ then \bar{g} has Hölder continuous derivatives and the proof of [Aub98] Proposition 1.25 shows we can additionally assume these are normal coordinates (i.e., the first derivatives of \bar{k} vanish at P). Finally, since the result we seek only involves properties of \bar{k} local to P , we can assume that $\bar{k} = 0$ except in a small coordinate ball B near P .

We claim there is a constant C such that

$$\int_B \frac{|\bar{k}|^p}{|x|^{2p}} \leq C \int_B |\bar{\partial}^2 \bar{k}|^p d\bar{V} \quad \text{and} \quad (5.8)$$

$$\int_B \frac{|\bar{\partial} \bar{k}|^p}{|x|^p} d\bar{V} \leq C \int_B |\bar{\partial}^2 \bar{k}|^p d\bar{V}. \quad (5.9)$$

Assuming for the moment that this claim is true, let $z_i = x_i/|x|^2$. Let $\bar{\phi} = |x|^{2-n}$ near P and extend $\bar{\phi}$ as a positive smooth function on the remainder of M . Let $g = \bar{\phi}^{N-2}/\bar{g}$. Near P , $\bar{\phi}^{N-2} = |x|^{-4}$ and so $g = e + k$ near infinity, where

$$k_{ij} := \bar{k}_{ij} - \frac{4}{|z|^2} z^a \bar{k}_{a(i} z_{j)} + \frac{4}{|z|^4} z^a z^b \bar{k}_{ab} z_i z_j = O(\bar{k}). \quad (5.10)$$

Since $k \in W_{\text{loc}}^{2,p}$, we need only establish the desired asymptotics at infinity.

A computation similar to the one leading to equation (5.4) shows

$$\begin{aligned} \partial k &= O(\bar{\partial} \bar{k}) O(|x|^2) + O(\bar{k}) O(|x|) \\ \partial^2 k &= O(\bar{\partial}^2 \bar{k}) O(|x|^4) + O(\bar{\partial} \bar{k}) O(|x|^3) + O(\bar{k}) O(|x|^2). \end{aligned} \quad (5.11)$$

Also, $d\bar{V} = |z|^{-2n} dV$ near P . Hence

$$\int |\partial^2 k|^p |z|^{4p-2n} dV = \int |\partial^2 k|^p |x|^{-4p} |x|^{2n} dV \quad (5.12)$$

$$= \int \left(O(\bar{\partial}^2 \bar{k}) \right)^p + \left(O(\bar{\partial} \bar{k}) O(|x|^{-1}) \right)^p + \left(O(\bar{k}) O(|x|^{-2}) \right)^p d\bar{V}. \quad (5.13)$$

From inequalities (5.8) and (5.9), quantity (5.13) is finite. Noting

$$4p - 2n = -n - \tau p + 2p \quad (5.14)$$

we conclude $|\partial^2 k| \in L_{\tau-2}^p$, as desired. A similar calculation shows that $|\partial k| \in L_{\tau-1}^p$ and $|k| \in L_{\tau}^p$. This concludes the proof, up to establishing inequalities (5.8) and (5.9).

Theorem 1.3 of [Bar86] implies that

$$\int_B \frac{|f|^p}{|x|^{2p}} d\bar{V} \leq c \int_B \frac{|\bar{\partial} f|^p}{|x|^p} d\bar{V} \leq C \int_B |\bar{\partial}^2 f|^p d\bar{V} < \infty \quad (5.15)$$

for smooth functions f that are compactly supported in B and vanish in a neighborhood of P . This inequality relies on the fact that $p \neq n$, which corresponds to the condition $\delta = 0$ in [Bar86] Theorem 1.3.

Let f_n be a sequence of smooth functions vanishing near P that converges to \bar{k} in $W^{2,p}$; such a sequence exists since $\bar{k} = 0$ at P , since $\bar{\partial} \bar{k} = 0$ at P if $p > n$, and since we have assumed that \bar{k} vanishes outside of B . By reduction to a subsequence we may assume that the values and first derivatives of sequence converge pointwise a.e., and using Fatou's Lemma we find

$$\begin{aligned} \int_B \frac{|\bar{k}|^p}{|x|^{2p}} &\leq \liminf_{n \rightarrow \infty} \int_B \frac{|f_n|^p}{|x|^{2p}} \\ &\leq C \lim_{n \rightarrow \infty} \int_B |\bar{\partial}^2 f_n|^p d\bar{V} \\ &= C \int_B |\bar{\partial}^2 \bar{k}|^p d\bar{V} < \infty. \end{aligned} \quad (5.16)$$

This is inequality (5.8), and a similar argument shows that inequality (5.9) holds as well. \square

The threshold $\tau = -2$ in Lemma 5.2 arises because there is a connection between the rate of decay of the AE metric and the rate of convergence of the metric at the point of compactification in a chosen coordinate system: roughly speaking, decay of order ρ^τ corresponds to convergence at a rate of $r^{-\tau}$. For a generic smooth metric we can use normal coordinates to obtain convergence at a rate of r^2 , but we cannot expect to do better generally. Hence the decompactification of a smooth metric will typically not decay faster than ρ^{-2} . Looking at the proof of Lemma 5.2, we note that it can be readily extended to $k > 2$ to show that a $W_\tau^{k,p}$ AE metric with $k \geq 2$, $p > n/k$ and $\tau = (n/p) - k$ can be compactified to a $W^{k,p}$ metric. But the decay condition $\tau = (n/p) - k$ is quite restrictive for $k > 2$: smooth metrics decompactify generally to metrics with decay $O(\rho^{-2})$, but compactification of a $W_{-2}^{k,p}$ metric would not be known to be C^3 , regardless of how high k and p are. A more refined analysis for $k > 2$ would need to take into account asymptotics of the Weyl or Cotton-York tensor, and we point to Herzlich [Her97] for related results in the C^k setting.

Proposition 5.3. *Let (M, g) and (\bar{M}, \bar{g}) be a pair of manifolds as in Lemma 5.2, related by $g = \bar{\phi}^{N-2}\bar{g}$. Then $y_g(M) = y_{\bar{g}}(\bar{M})$.*

Proof. For simplicity we assume that M has one end. Let $P \in \bar{M}$ be the singular point of $\bar{\phi}$. Note that $W_c^{1,2}(M)$ is dense in $W_{\delta^*}^{1,2}(M)$ and that

$$S_P := W^{1,2}(\bar{M}) \cap \{u : u|_{B_r(P)} = 0 \text{ for some } r > 0\} \quad (5.17)$$

is dense in $W^{1,2}(\bar{M})$ since $2 < n$. From upper semicontinuity of the Yamabe quotient, the Yamabe invariants of g and \bar{g} can be computed by minimizing the Yamabe quotient over $W_c^{1,2}$ and S_P respectively. Note that $u \mapsto \bar{\phi}u$ is a bijection between $W_c^{1,2}(M)$ and S_P . The proof of Lemma 3.7 shows that for $u \in W_c^{1,2}$,

$$Q_g^y(u) = Q_{\bar{g}}^y(\bar{\phi}u) \quad (5.18)$$

and hence $y_g(M) = y_{\bar{g}}(\bar{M})$. \square

Combining Lemma 5.2 and Proposition 5.3 we obtain our second classification.

Proposition 5.4. *Let (M, g) be a $W_\tau^{2,p}$ AE manifold with $\tau \leq \frac{n}{p} - 2$. Then (M, g) is Yamabe positive/negative/null if and only if some conformal compactification, as described in Lemma 5.2, has the same Yamabe type.*

Consequently, Yamabe classification on AE manifolds has the same topological flavor as in the compact setting. For instance, since the torus does not allow a Yamabe positive metric, the decompactified torus, which is diffeomorphic to \mathbb{R}^n with a handle, does not allow a metric with nonnegative scalar curvature.

We mention an application of Proposition 5.4 to general relativity. In general relativity, spacetimes can be constructed by specifying initial data in the form of a Riemannian manifold (M, g) and a symmetric (0,2)-tensor K , and then solving a hyperbolic evolution problem to construct an ambient Lorentzian spacetime such that g and K are the induced metric and second fundamental form of the initial hypersurface. However, the initial data cannot be freely specified; it must satisfy the Einstein constraint equations,

$$\begin{aligned} R - |K|^2 + \text{tr} K^2 &= r, \\ \text{div} K - d \text{tr} K &= j, \end{aligned} \quad (5.19)$$

where r is the energy density and j is the momentum density of matter. It is natural to suppose that the energy density r is everywhere nonnegative, which is known as the weak energy condition. If the initial data

is maximal, i.e., if the mean curvature $\text{tr } K$ is zero, then the weak energy condition implies $R \geq 0$. Thus, if the compactification of an AE manifold has a topology that does not admit a Yamabe positive metric, then the original AE manifold does not allow maximal initial data satisfying the weak energy condition.

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References

- [Aub76] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.*, 55:269–296, 1976.
- [Aub98] T. Aubin. *Some Nonlinear Problems in Riemannian Geometry*. Springer, 1998.
- [Bar86] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.
- [BE87] J. Bourguignon and J. Ezin. Scalar Curvature Functions in a Conformal Class of Metrics and Conformal Transformations. *Transactions of the American Mathematical Society*, 301(2):723–736, June 1987.
- [CB81] M. Cantor and D. Brill. The Laplacian on asymptotically flat manifolds and the specification of scalar curvature. *Compositio Math.*, 43(3):317–330, 1981.
- [DGI15] J. Dilts, R. Gicquaud, and J. Isenberg. A nonexistence result for the conformal constraint equations on asymptotically Euclidean manifolds. *Preprint*, 2015.
- [ES86] J. Escobar and R. Schoen. Conformal metrics with prescribed scalar curvature. *Invent. Math.*, 86(2):243–254, 1986.
- [FCS80] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33(2):199–211, 1980.
- [Fri11] H. Friedrich. Yamabe numbers and the Brill-Cantor criterion. *Ann. Henri Poincaré*, 12(5):1019–1025, 2011.
- [Her97] M. Herzlich. Compactification conforme des variétés asymptotiquement plates. *Bull. Soc. Math. France*, 125(1):55–91, 1997.
- [LP87] J. Lee and T. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.
- [Max05a] D. Maxwell. Rough solutions of the Einstein constraint equations on compact manifolds. *J. Hyperbolic Differ. Equ.*, 2(2):521–546, 2005.

- [Max05b] D. Maxwell. Solutions of the Einstein constraint equations with apparent horizon boundaries. *Comm. Math. Phys.*, 253(3):561–583, 2005.
- [Rau95] A. Rauzy. Courbures scalaires des variétés d’invariant conforme négatif. *Trans. Amer. Math. Soc.*, 347(12):4729–4745, 1995.
- [Sch84] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Diff. Geom.*, 20:479–495, 1984.
- [Tri76a] H. Triebel. Spaces of Kudrajavcev type. I. Interpolation, embedding, and structure. *J. Math. Anal. Appl.*, 56(2):253–277, 1976.
- [Tri76b] H. Triebel. Spaces of Kudrajavcev type. II. Spaces of distributions: duality, interpolation. *J. Math. Anal. Appl.*, 56(2):278–287, 1976.
- [Tru68] N. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa*, 22:265–274, 1968.
- [Tru73] N. Trudinger. Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa (3)*, 27:265–308, 1973.
- [Yam60] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.